Cayley Olympiad 2018 Solutions

C1. The positive integer N has six digits in increasing order. For example, 124 689 is such a number.

However, unlike 124 689, three of the digits of N are 3, 4 and 5, and N is a multiple of 6.

How many possible six-digit integers N are there?

Solution

The digits of N increase, so the known, consecutive digits 3, 4 and 5 must be adjacent and in increasing order.

Since N has six digits, and they increase, the final digit must be at least 5.

Since N is a multiple of 6, it is even, so the final digit is even and must be 6 or 8. This means that 3 (as the start of 345) must be one of the first three digits of N.

This gives the following possible values of N:

345678, 134568, 134578, 234568, 234578, 123456 and 123458

Since N is a multiple of 3, the sum of the digits of N must be a multiple of 3.

The possibilities 134578, 234568, 234578 and 123458 have digit sums 28, 28, 29 and 23 respectively so cannot be values for N.

But 345678, 134568, 123456 have digit sums 33, 27 and 21 respectively.

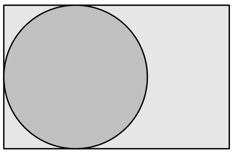
Therefore these are the three possible six-digit integers N.

[An alternative interpretation of the question allows digits in the solutions to repeat, since such a number would have digits that are in increasing order, even though the digits themselves do not always increase. This interpretation yields 19 solutions.]

C2. A circle lies within a rectangle and touches three of its edges, as shown.

The area inside the circle equals the area inside the rectangle but outside the circle.

What is the ratio of the length of the rectangle to its width?



Solution

Let the radius of the circle be r and the length of the rectangle be s.

Therefore the width of the rectangle is $2r$.

The area inside the circle is πr^2.

The area inside the rectangle but outside the circle is $2rs - \pi r^2$.

These quantities are equal so

$$\pi r^2 = 2rs - \pi r^2.$$

Adding πr^2 to each side gives

$$2\pi r^2 = 2rs.$$

Since r is not zero, division by $2r$ is permissible and gives

$$\pi r = s.$$

Therefore the ratio of the length of the rectangle to the width is $\pi r : 2r$, which simplifies to $\pi : 2$.

C3. The addition sum $XCV + XXV = CXX$ is true in Roman numerals.

In this question, however, the sum is actually the letter-sum shown alongside, in which: each letter stands for one of the digits 0 to 9, and stands for the same digit each time it occurs; different letters stand for different digits; and no number starts with a zero.

$$\begin{array}{r} XCV \\ +\,XXV \\ \hline CXX \end{array}$$

Find all solutions, and explain how you can be sure you have found every solution.

Solution

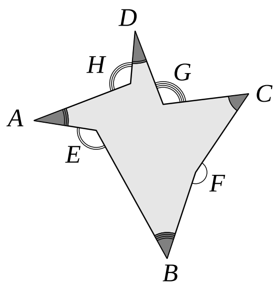
First, since C is the leading digit of a three-digit number, $C > 0$.

From the tens column we see that $C + X$ results in an X. Since C cannot be 0, there must be a carry digit of 1 from the units column, and $C + 1 = 10$. Therefore $C = 9$ and V is at least 5.

Now from the hundreds column $X = 4$, so from the units column $V = 7$.

Thus the unique solution is $497 + 447 = 944$.

C4. Prove that the difference between the sum of the four marked interior angles A, B, C, D and the sum of the four marked exterior angles E, F, G, H of the polygon shown is 360°.



Solution

Let the sum of the four marked interior angles be S.

The four remaining interior angles are $360 - E$, $360 - F$, $360 - G$ and $360 - H$ since the angles at a point add up to 360 degrees.

The polygon has eight sides so its interior angles add up to 1080 degrees. Hence $1080 = S + 360 - E + 360 - F + 360 - G + 360 - H$.

This equation simplifies to $1080 = 1440 + S - (E + F + G + H)$, which is equivalent to $E + F + G + H - S = 360$.

Hence the difference between the sum of the four marked interior angles and the four marked exterior angles is 360 degrees, as required.

C5. In the expression below, three of the + signs are changed into − signs so that the expression is equal to 100:

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$+ 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

In how many ways can this be done?

Solution

The given expression is equal to 210.

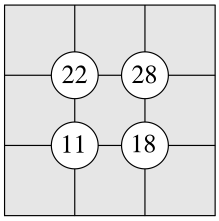
Changing three signs from + to – will decrease the expression by double the sum of the three affected terms. So the sum of the three terms must be half of 210 – 100, which is 55.

The three terms must be distinct integers that are less than or equal to 20. The largest possible total for three such terms is $18 + 19 + 20$, which is 57. To achieve a total of 55, we need to decrease the sum of the three terms by 2. This can be done by reducing the 18 by 2 or reducing the 19 by 2 (which is equivalent to taking one off each of the 18 and 19).

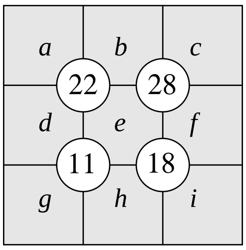
Hence, the expression can be changed to give a total of 100 in two ways – either by changing the signs on the 16, 19 and 20 or on the 17, 18 and 20.

C6. In the puzzle *Suko*, the numbers from 1 to 9 are to be placed in the spaces (one number in each) so that the number in each circle is equal to the sum of the numbers in the four surrounding spaces.

How many solutions are there to the *Suko* puzzle shown alongside?



Solution



Using the labelling shown in the diagram, $d + e + g + h = 11$ where d, e, g and h are distinct positive integers. The smallest four positive distinct integers (1, 2, 3 and 4) have total 10. To give a total of 11, the only integer that can increase by one and remain distinct is the 4, which means that d, e, g and h must be 1, 2, 3 and 5, in some order.

Similarly, $b + c + e + f = 28$, which means that b, c, e and f must be 5, 6, 8 and 9, in some order, or 4, 7, 8 and 9 in some order.

Since e is the only variable in both lists and 5 is the only number in both lists, e must be 5, and b, c and f must be 6, 8 and 9 in some order.

The only numbers from 1 to 9 not accounted for so far are 4 and 7 so these must be the values of a and i, in some order.

If $a = 4$ and $i = 7$, consider the value of f. Since b, c, e and f must be 5, 6, 8 and 9, in some order, and $e \neq f$, f must be at least 6.

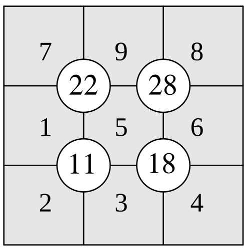
But $e + f + h + i = 5 + f + h + 7 = 18$, which would make h less than or equal to 0, which is impossible.

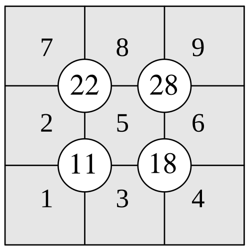
Hence $a = 7$ and $i = 4$.

Consider the value of f. If $f = 9$, $e + f + h + i = 5 + 9 + h + 4 = 18$, so $h = 0$, which is impossible.

If $f = 8$, it follows that $h = 1$ and $d = 2$ or 3, and hence $g = 3$ or 2 respectively. This would mean $b = 8$ or 7 respectively, both of which are impossible.

If $f = 6$, it follows that $h = 3$ and $d = 1$ or 2, and hence $g = 2$ or 1 respectively. This would mean $b = 9$ or 8, respectively. These values are possible and give the complete solutions:





There are two solutions.